Model Answer

Differential Geometry (AV-8206)

M.A./M. Sc. (First Semester) Examination 2015-16

1.(i) Normal plane: The plane which containing principal normal and binormal is called normal plane.
Rectifying Plane: The plane which containing tangent and binormal is called rectifying plane.

(ii) Serret Frenet Formulae: The relations

$$t^{i\prime} = K p^i, \quad p^{i\prime} = \tau b^i - K t^i, \quad b^{i\prime} = -\tau p^i$$

are known as Serret-Frenet formulae.

(iii) Arc length: The arc length of a curve $x^i = x^i(t)$ from the point of parameter to to a variable point of parameter t is defined by

$$s = \int_{t_0}^{t} \left(\frac{dx^i}{dt} \cdot \frac{dx^i}{dt} \right)^{1/2} dt.$$

(iv) Given that

$$x^i = (3\cosh 2u, 3\sinh 2u, 6u)$$

$$\frac{dx^i}{du} = (6\sinh 2u, 6\cosh 2u, 6)$$

$$\therefore \quad s = \int_0^5 \left[(6\sinh 2u, 6\cosh 2u, 6) \cdot (6\sinh 2u, 6\cosh 2u, 6) \right]^{1/2} du$$

$$= 6\int_0^5 \left[1 + \sinh^2 2u + \cosh^2 2u \right]^{1/2} du$$

$$= 6\sqrt{2} \int_0^5 \cosh 2u \, du = \frac{6\sqrt{2}}{2} \left[\sinh 10 \right]$$

$$= \underline{3\sqrt{2}\,\sinh 10}. \text{ Ans.}$$

(iv) Mean Curvature: Let K_1 and K_2 be two principal curvatures at a point of surface then mean curvature is defined by $M = K_1 + K_2$

(iv) Gaussian Curvature: Gaussian curvature is defined as

$$G = K_1 \cdot K_2 .$$

(v) Given that $\bar{r} = (u+v, u-v, u)$

$$x = u+v ; \; y = u-v, \; z = u$$

$$\Rightarrow \quad x + y = 2z$$

(vi) Dupin's Theorem: The sum of normal curvatures in two orthogonal direction is constant and equal to the sum of the principal curvatures.

(vii). Umbilical point: umbilical point is a point at which the principal radii are equal.

(viii) Evolute: If $\bar{C}$ is an involute of a curve C, then C is called an evolute of the curve $\bar{C}$.

Equation of evolute is given by

$$x^i = x^i + \rho \left\{ p^i + \cot\left(\int \tau \, ds + c\right) b^i \right\}.$$

(ix) Given $x = (u^1 \cos u^2, u^1 \sin u^2, c u^2)$

$$x_1 = (\cos u^2, \sin u^2, 0)$$
$$x_2 = (-u^1 \sin u^2, u^1 \cos u^2, c)$$

$$g_{11} = 1, \; g_{12} = 0, \; g_{22} = (u^1)^2 + c^2$$
$$ds^2 = (du^1)^2 + [(u^1)^2 + c^2] (du^2)^2 .$$

(x) Equation of the curve $x^i = x^i(t)$.

curvature K and binormal are given by

$$b^i = \frac{\rho}{\dot{\phi}^3}\left(\dot{x}^j \ddot{x}^k - \dot{x}^k \ddot{x}^j\right)$$

$$K = \frac{\sqrt{\ddot{x}^i \ddot{x}^i - \ddot{\phi}^2}}{\dot{\phi}^3}$$

2. Given surface is

$$x^2 + z^2 = y \quad \text{———} \quad (I)$$

Curve is

$$x^1 = t,\ x^2 = t^2,\ x^3 = 3t^3 \quad \text{———} \quad (II)$$

For point of intersection of (I) and (II), we have

$$\cancel{F(t) = t^4 + 9t^6 - t}$$

$$F(t) \equiv t^2 + 9t^6 - t^2$$

$$\Rightarrow \quad F(t) \equiv 9t^6, \quad \text{at origin } t = 0$$

it is clear that

$$\left(\frac{dF}{dt}\right) = \left(\frac{d^2F}{dt^2}\right) = \left(\frac{d^3F}{dt^3}\right) \cdots = \left(\frac{d^5F}{dt^5}\right) = 0$$

and $\left(\frac{d^6F}{dt^6}\right)_{at\ t=0} \neq 0$

Hence (I) and (II) have six point contact at origin.

3. Osculating sphere: The osculating sphere at a point P on the curve is a sphere which has four point contact with the curve at P.

Osculating plane: The osculating plane of a curve at a point P is the limiting position of the plane determined by the tangent at P and a point Q of the curve as Q approaches P along the curve.

Osculating Circle: The intersection of osculating sphere and osculating plane is called osculating circle.

Given that $\bar{r}(t) = (\cos t, a\sin t, ct)$

We know that osculating plane (equation) is given by

$$\begin{vmatrix} X^1 - x^1, & X^2 - x^2, & X^3 - x^3 \\ \dot{x}^1 & \dot{x}^2 & \dot{x}^3 \\ \ddot{x}^1 & \ddot{x}^2 & \ddot{x}^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} X^1 - \cos t & X^2 - a\sin t & X^3 - ct \\ -\sin t & a\cos t & c \\ -\cos t & -a\sin t & 0 \end{vmatrix} = 0$$

required equation.

4. Given that

$$x^3 = c \tan^{-1}\left(\frac{x^2}{x^1}\right) \qquad \text{---} \ (I)$$

Parametric equation of the above surface is given by

$$x^1 = u\cos\theta, \quad x^2 = u\sin\theta, \quad x^3 = c\theta$$

$$x = (u\cos\theta,\ u\sin\theta,\ c\theta)$$

Now calculate g^{11}, g^{12}, g^{22}, d_{11}, d_{12}, d_{22}, d, and g. Putting these values in

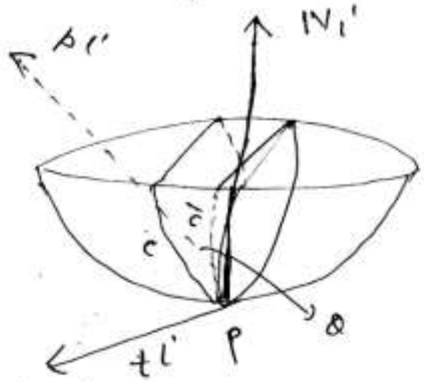
$$K_n^2 - K_n\left(g^{\alpha\beta} d_{\alpha\beta}\right) + \frac{d}{g} = 0 \qquad \text{---} \ (II)$$

We get $\rho_1 = \dfrac{u^2+c^2}{c}$, $\rho_2 = -\dfrac{u^2+c^2}{c}$

$$\Rightarrow \quad \rho_1 + \rho_2 = 0.$$

5. Meunier's theorem: If K and K_n denote the curvature of oblique and normal sections through the same tangent line, and θ be the angle between the sections,

$$K_n = K\cos\theta.$$



Proof: Let the oblique section and normal section at P of the surface through the same tangent line cuts the surface along the curves C and $\bar{C}$ respectively.

Also p^i be the principal normal of the curve C at P (6)
and N^i be the normal to the surface. By Serret Frenet formula, we have

$$x''^i = Kp^i \quad \text{—— (I)}$$

Since θ is the angle between N^i and p^i, we have

$$N^i . p^i = \cos\theta . \quad \text{—— (II)}$$

(∴) Multiplying (I) by N^i and using (II), we have.

$$K \cos\theta = N^i . x''^i$$

$$\Rightarrow k_n \cos\theta = N^i . \frac{d^2x^i}{ds^2}$$

$$\Rightarrow \boxed{K = K_n \cos\theta}.$$

6 Given Curve is

$$\bar{r} = (3u, 3u^2, 2u^3). \quad \text{—— (I)}$$

We know that equation of ~~e~~ involute is given by

$$x^i = x^i + (c-s)\, t^i \quad \text{—— (ii)}$$

From (I)

$$t^i = (3, 6u, 6u^2)\frac{1}{\sqrt{9+36u^2+36u^4}}$$

$$t^i = (1, 2u, 2u^2)\frac{1}{\sqrt{1+4u^2+4u^4}}$$

$$= (1, 2u, 2u^2)\frac{1}{1+2u^2}.$$

Now $s = \int \left(\frac{dx^i}{du} \frac{dx^i}{du} \right)^{1/2} du$

$$= \int (1 + 2u^2)\, du$$

putting these values in (II), we get the required equation.

7. Let K_1 and τ_1 be the curvature and torsion of the spherical indicatrix of binormal. We know that For spherical indicatrix of binormal

$$\bar{r}_1 = \bar{b} \quad \text{———} \quad (I)$$

differentiating we get

$$\frac{d\bar{r}_1}{ds_1} = \frac{d\bar{b}}{ds} \cdot \frac{ds}{ds_1} \quad \text{———} \quad (II)$$

$$\Rightarrow \quad \bar{t}_1 = \frac{d\bar{b}}{ds} \cdot \frac{ds}{ds_1} = -\tau \bar{p} \frac{ds}{ds_1} \quad \text{———} \quad (III)$$

$$\Rightarrow \quad \left| \frac{ds}{ds_1} \right| = 1/\tau \quad \text{———} \quad (IV)$$

From (III) and (IV)

$$\bar{t}_1 = -\bar{p} \quad \text{———} \quad (V)$$

again differentiating

$$K_1 \bar{p}_1 = -(\tau \bar{b} - K t) \cdot \frac{1}{\tau} \quad \text{———} \quad (VI)$$

Taking modulus of both side, we have.

$$K_1 = \frac{\sqrt{K^2 + \tau^2}}{\tau}$$

Similarly differentiating (vi) and using above results, we have. (8)

$$\gamma_1 = \frac{k'\gamma' - k\gamma'}{\gamma\,(k^2+\gamma^2)}$$

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8 Given that

$$x^1 = a(3t-t^3),\ x^2 = 3at^2,\ x^3 = a(3t+t^3).$$

We know that For curvature.

$$K = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} \quad \text{———— (1)}$$

Here

$$\bar{r} = \left(a(3t-t^3),\ 3at^2,\ a(3t+t^3)\right)$$

Find $\dot{\bar{r}}$, $\ddot{\bar{r}}$ and putting in (1), we get

$$K = \frac{1}{3a(1+t^2)^2}.$$

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